Self-similar collapse of a massless scalar field in three-dimensions.

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Abstract

We study an analytical solution to the Einstein's equations in $2 + 1$-dimensions, representing the self-similar collapse of a circularly symmetric, minimally coupled, massless, scalar field. Depending on the value of certain parameters, this solution represents the formation of black holes. Since our solution is asymptotically flat, our black holes do not have the BTZ space-time as their long time limit. They represent a new family of black holes in $2 + 1$-dimensions.

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Since the work of M. W. Choptuik on the gravitational collapse of a massless scalar field Ref. [1], many physicists have focused their attentions on the issue of gravitational collapse. In that work Choptuik showed that one may understand the gravitational collapse as a critical phenomena. In a sense that there is a critical value for a certain parameter, in the parameter space of solutions, which separates two types of solutions: those which are black holes from others which are not black holes.

An important arena where one can study the gravitational collapse is general relativity in $2 + 1$-dimensions. The great appeal of this theory comes from the fact that it retains many of the properties of general relativity in $3 + 1$-dimensions, but the field equations are greatly simplified [2].

Presently, several black hole solutions in $2 + 1$-dimensional general relativity are known [3]. Including the first one to be discovered, the so-called BTZ black hole [4]. All of them have an important property in common: the presence of a negative cosmological constant, which makes them asymptotically anti-de Sitter.

Indeed, in a recent work it was demonstrated that a three-dimensional solution to the Einstein's equations, with a positive cosmological constant (Λ), such that the stress-energy tensor satisfies the dominant energy condition, contains no apparent horizons [5]. The same result applies to the case $\Lambda = 0$ in the presence of matter fields. Therefore, this result explains the necessity of a negative cosmological constant in order to a black hole to form, in three-dimensional general relativity.

Considering the conditions used to derive the, above mentioned, theorem it might be possible to have the formation of an apparent horizon in a space-time in three-dimensions, without a negative cosmological constant. For this, the matter content of this space-time should have a stress-tensor that does not satisfy the dominant energy condition. It is this possibility that we shall investigate here. We shall allow the, initially real, scalar field to become complex which will force its stress-energy tensor to violate the dominant energy condition. We are aware that there is a great debate whether this matter content is physically acceptable or not [6], but the theoretical possibility of a new family of black holes in $2 + 1$-

dimensions, we believe, is enough motivation to use it here.

In the present letter we would like to present a solution to the Einstein's equation, without a cosmological constant, representing the self-similar, circularly symmetric, collapse of a minimally coupled, massless, scalar field, in $2 + 1$-dimensions. As we shall see this solution, depending on the value of certain parameters, represents the formation of black holes as the result of the collapse process.

We shall start by writing down the ansatz for the space-time metric. As we have mentioned before, we would like to consider the circularly symmetric, self-similar, collapse of a massless scalar field in $2 + 1$-dimensions. Therefore, we shall write our metric ansatz as,

$$ds^2 = -2e^{2\sigma(u,v)}dudv + r^2(u, v)d\theta^2 \,, \tag{1}$$

where $\sigma(u, v)$ and $r(u, v)$ are two arbitrary functions to be determined by the field equations, (u, v) is a pair of null coordinates varying in the range $(-\infty, \infty)$, and θ is an angular coordinate taking values in the usual domain $[0, 2\pi]$.

The scalar field Φ will be a function only of the two null coordinates and the expression for its stress-energy tensor $T_{\alpha\beta}$ is given by [7],

$$T_{\alpha\beta} = \Phi_{,\alpha}\,\Phi_{,\beta} - \frac{1}{2}g_{\alpha\beta}\Phi_{,\lambda}\,\Phi^{,\lambda}\,. \tag{2}$$

where , denotes partial differentiation.

Now, combining Eqs. (1) and (2) we may obtain the Einstein's equations which in the units of Ref. [7] and after re-scaling the scalar field, so that it absorbs the appropriate numerical factor, take the following form,

$$2\sigma_{,u}\,r_{,u} - r_{,uu} = r(\Phi_{,u})^2\,, \tag{3}$$

$$2\sigma_{,v}\,r_{,v} - r_{,vv} = r(\Phi_{,v})^2\,, \tag{4}$$

$$2r\sigma_{,uv} + r_{uv} = -r(\Phi_{,u}\,\Phi_{,v})\,, \tag{5}$$

$$r_{,uv} = 0,\qquad(6)$$

The equation of motion for the scalar field, in these coordinates, is

$$2r\Phi_{,uv} + \Phi_{,v}\,r_{,u} + \Phi_{,u}\,r_{,v} = 0.\qquad(7)$$

The above system of non-linear, second-order, coupled, partial differential equations (3)-(7) has an analytical solution if we impose that it is continuously self-similar. More precisely, following Coley [8], our system will have self-similarities of the first and second kinds.

Under these conditions our solution will be given by,

$$r(u,v) = \beta(\alpha v)^{1/\alpha} + \gamma u,\qquad(8)$$

$$\sigma(u,v) = \left(\frac{1-\alpha}{2}\right)\ln\left(\frac{r}{u}\right) + \sigma_0,\qquad(9)$$

and the scalar field has the following value,

$$\Phi(u,v) = (1-\alpha)^{1/2}\ln\left[\frac{\sqrt{(\gamma/\beta)u} - \imath\sqrt{(\alpha v)^{1/\alpha}}}{\sqrt{(\gamma/\beta)u} + \imath\sqrt{(\alpha v)^{1/\alpha}}}\right],\qquad(10)$$

where γ and β are real, integration constants and α is a positive real number associated with the kinematic, continuous, self-similarity. For $\alpha = 1$, the self-similarity is of the first kind, for $0 < \alpha < 1$ the self-similarity is of the second kind [8]. Following [9], we shall assume that $\Phi(u,v) \equiv 0$ for $v < 0$.

In terms of $r(u,v)$ Eq. (8), and $\sigma(u,v)$ Eq. (9), the line element Eq. (1) becomes,

$$ds^2 = -2e^{2\sigma_0}\left(\frac{r}{u}\right)^{(1-\alpha)}dudv + r^2d\theta^2.\qquad(11)$$

One may notice from Eqs. (8-10), that for different values of α, β and γ, one has different space-times.

Observing Eq. (11), we notice that these space-times have a singularity at $r = 0$. It is a physical singularity as can be seen directly from the curvature scalar R.

In order to show this result we start writing down the Ricci tensor that, in the present case, has the following expression [10],

$$R_{\alpha\beta} = \Phi_{,\alpha} \, \Phi_{,\beta} \ . \tag{12}$$

From it, we may compute R straightforwardly with the aid of Eqs. (8)-(11), finding,

$$R = -2(1-\alpha)\gamma\beta e^{-2\sigma_0} \frac{[(\alpha v)^{1/\alpha} u]^{(1-\alpha)}}{r^{(3-\alpha)}} \ . \tag{13}$$

Finally, taking the limit $r \to 0$ in R Eq. (13), we find that this quantity diverges at $r = 0$. There is no other physical singularity for these space-times because R is well defined outside $r = 0$. In particular, $u = 0$ is just an apparent singularity and a new coordinate system can be found where it disappears. As we shall see below, $u = 0$ is an apparent horizon for these space-times.

Another important property we can learn from R is the asymptotic behavior of our solution. If we take the limit $r \to \infty$ of R Eq. (13), we find that $R \to 0$. Therefore, we conclude that the space-times under investigation are asymptotically flat.

The apparent horizons are determined by imposing that the surface $r = $ constant becomes null, which implies that,

$$2g^{uv} \, r_{,u} \, r_{,v} = 0 \ . \tag{14}$$

For the above space-times, this equation (14) takes the following form when we introduce the appropriate information from Eqs. (8) and (11),

$$- 2e^{-2\sigma_0} \gamma\beta \left[\frac{(\alpha v)^{1/\alpha} u}{r} \right]^{(1-\alpha)} = 0 \ . \tag{15}$$

In the most general case, the space-times under study may have two distinct apparent horizons, from the solutions of Eq. (15). They are the surfaces $u = 0$ and $v = 0$.

The space-times above will only be physically relevant for the collapse process if r Eq. (8) is a real, positive function. Also, at least outside the horizon, $r = $ constant, must be a set of time-like surfaces for different constants. On the other hand, as we have mentioned above the scalar field should be allowed to take complex values if we want to obtain black hole solutions.

It is clear from Eqs. (11) and (13) that $\alpha = 1$ is the three-dimensional Minkowski space-time. Therefore, we shall restrict our attention to the space-times with self-similarity of the second kind. From Eq. (10), it is appropriate to consider the influx of scalar field to be turned on at the advanced time $v = 0$. So that to the past of this surface the space-time is Minkowskian and the metric is therefore C^1 at this surface.

The black hole space-times are obtained for $\gamma < 0$ and $\beta > 0$. Figure 1 shows a conformal diagram for a typical space-time in this case. We may see that the space-time is divided, naturally, in three distinct regions. The first one is the Minkowskian region where $v < 0$ (I). Then, we have the external region where $v > 0$ and $u < 0$ (II). Finally, in the internal region $u > 0$ (III).

The scalar field starts collapsing from past null infinity, in the external region. From eq. (10), it is not difficult to see that it is imaginary in this region. It is zero at $v = 0$, grows to $\Phi = \imath(\pi/2)\sqrt{1-\alpha}$ at $u = (\beta/\gamma)(\alpha v)^{1/\alpha}$ and reaches its maximum value, in this region, $\Phi = \imath\pi\sqrt{1-\alpha}$, at the apparent horizon $u = 0$. In the internal region we may re-write $\Phi(u,v)$ Eq. (10) as,

$$\Phi(u,v) = \sqrt{1-\alpha}[\ln r \; - \; 2\ln\left(\sqrt{|\gamma|u} + \sqrt{\beta(\alpha v)^{1/\alpha}}\right) + \imath\pi]. \tag{16}$$

From it we see that $\Phi(u,v)$ becomes complex with a constant imaginary part. Its real part decreases from zero at the horizon $u = 0$, until it blows up at the singularity $r = 0$.

Observing Eqs.(14) and (15), we see that the surfaces $r =$ constant will be time-like in both external and internal regions if $1 - \alpha = l/m$, where l and m are integer numbers, l being even and m being odd. Therefore, for this choice of α, the singularity $r = 0$ will be a time-like one. Once that this singularity is hidden by the horizon $u = 0$, we may consider these space-times representing black holes.

Since our solutions are asymptotically flat, our black holes do not have the BTZ space-time as their long time limit. They represent a new family of black holes in $2+1$-dimensions.

We may also describe our solution with the aid of the time coordinate,

$$t(u,v) = \beta(\alpha v)^{1/\alpha} - \gamma u, \tag{17}$$

in terms of which, the line element Eq. (11) and the scalar field Eq. (10) become, respectively,

$$ds^2 = -\frac{(2)^{(1-2\alpha)}e^{2\sigma_0}}{(\beta\gamma)^\alpha}\left(\frac{r}{r^2-t^2}\right)^{(1-\alpha)}(-dt^2+dr^2)+r^2d\theta^2 \qquad (18)$$

and

$$\Phi(r,t) = \sqrt{1-\alpha}\,\ln\left[\frac{\sqrt{r-t}-\imath\sqrt{r+t}}{\sqrt{r-t}+\imath\sqrt{r+t}}\right]. \qquad (19)$$

We end the letter by noting that recently another continuously self-similar solution, to the same problem treated here, was found [11]. It is not difficult to realize that the solution described here is different from the one derived in [11], because the latter has a self-similarity of the first kind [8]. As we have seen, if we set the condition that our solution has just a self-similarity of the first kind ($\alpha = 1$), it reduces to Minkowski space-time. Therefore, they are not the same solution.
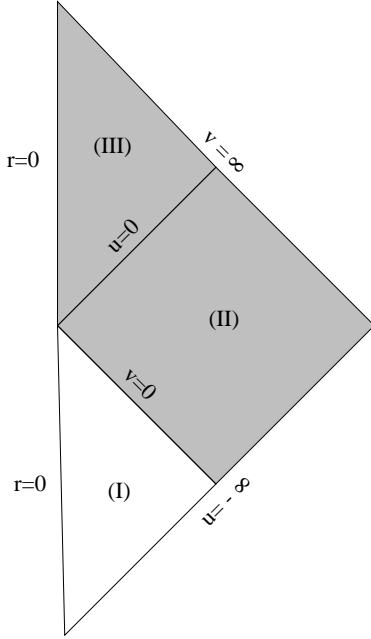
FIG. 1. Conformal diagram for a typical black hole solution. The apparent horizon $u = 0$ separates the Minkowskian (I) and external (II) regions from the internal region (III) where lies the time-like singularity at $r = 0$.

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